

# LOEWNER THEORY FOR QUASICONFORMAL EXTENSIONS: OLD AND NEW

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**ABSTRACT.** This survey article gives an account of quasiconformal extensions of univalent functions with its motivational background from Teichmüller theory and classical and modern approaches based on Loewner theory.

## 1. UNIVERSAL TEICHMÜLLER SPACES

The notion of the universal Teichmüller spaces was illuminated in the theory of quasiconformal mappings as an embedding of the Teichmüller spaces of compact Riemann surfaces of finite genus. Several equivalent models of universal Teichmüller spaces are known (see e.g. [Sug07]). In this note we would like to focus on the connection with a space of the Schwarzian derivatives of conformal extensions of quasiconformal mappings defined on the upper half-plane  $\mathbb{H}^+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$ .

**1.1. Quasiconformal mappings.** A homeomorphism  $f$  of a domain  $G \subset \mathbb{C}$  is called  **$k$ -quasiconformal** if  $f_z$  and  $f_{\bar{z}}$ , the partial derivatives in  $z$  and  $\bar{z}$  in the distributional sense, are locally integrable on  $G$  and satisfy

$$|f_{\bar{z}}(z)| \leq k|f_z(z)| \quad (1.1)$$

almost everywhere in  $G$ , where  $k \in [0, 1)$ . Let  $B(G)$  be the open unit ball  $\{\mu \in L^\infty(G) : \|\mu\|_\infty < 1\}$  of  $L^\infty(G)$ , where  $L^\infty(G)$  is a complex Banach space of all bounded measurable functions on  $G$ , and  $\|\mu\|_\infty := \text{ess sup}_{z \in G} |\mu(z)|$  for a  $\mu \in L^\infty(G)$ . An element  $\mu \in B(G)$  is called the **Beltrami coefficient**. If  $f$  is a  $k$ -quasiconformal mapping on  $G$ , then it is verified that  $f_z(z) \neq 0$  for almost all  $z \in G$ . Hence  $\mu_f := f_{\bar{z}}/f_z$  defines a function belongs to  $B(G)$ .  $\mu_f$  is called the **complex dilatation** of  $f$ , and the maximal of  $k := k(f) := \|\mu_f\|_\infty$  is called the **maximal dilatation** of  $f$ . Conversely, the following fundamental existence and uniqueness theorem is known.

**Theorem 1.1** (The measurable Riemann mapping theorem). *For a given measurable function  $\mu \in B(\mathbb{C})$ , there exists a unique solution  $f$  of the equation*

$$f_{\bar{z}} = \mu f_z \quad (1.2)$$

*for which  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a quasiconformal mapping fixing the points 0 and 1.*

The equation (1.2) is called the **Beltrami equation**.

The reader is referred to [Ahl06], [LV73] and [IT92] for the general theory of quasiconformal mappings in the plane. Here we recall some fundamental properties of quasiconformal mappings we will use later.

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$f$  is 0-quasiconformal if and only if  $f$  is conformal. If  $f$  is  $k$ -quasiconformal, then so is its inverse  $f^{-1}$  as well. A composition of a  $k_1$ - and  $k_2$ -quasiconformal map is  $(k_1 + k_2)/(1 + k_1 k_2)$ -quasiconformal. The composition property of the complex dilatation is the following; Let  $f$  and  $g$  be quasiconformal maps on  $G$ . Then the complex dilatation  $\mu_{g \circ f^{-1}}$  of the map  $g \circ f^{-1}$  is given by

$$\mu_{g \circ f^{-1}}(f) = \frac{f_z}{f_{\bar{z}}} \cdot \frac{\mu_g - \mu_f}{1 - \mu_g \mu_f}. \quad (1.3)$$

Since a 0-quasiconformal map is conformal, the above formula concludes that if  $\mu_f = \mu_g$  almost everywhere in  $G$  then  $g \circ f^{-1}$  is conformal on  $f(G)$ .

As the case of conformal mappings, isolated boundary points of a domain  $G$  are removable singularities of every quasiconformal mapping of  $G$ . It follows from this property that quasiconformal and conformal mappings divide simply-connected domains into the same equivalence classes.

**1.2. Schwarzian derivatives.** Let  $f$  be a non-constant meromorphic function with  $f' \neq 0$ . Then we define the **Schwarzian derivative** by means of

$$S_f := \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$

It is known that  $f$  is a Möbius transformation if and only if  $S_f \equiv 0$ . Further, a direct calculation shows that

$$S_{f \circ g} = (S_f \circ g)g'^2 + S_g.$$

Hence it follows the invariance property of  $S_f$  that if  $f$  is a Möbius transformation then  $S_{f \circ g} = S_g$ . One can interpret that the Schwarzian derivative measures the deviation of  $f$  from a Möbius transformation. In order to describe it precisely, we introduce the norm of the Schwarzian derivative  $\|S_f\|_G$  of a function  $f$  on  $G$  by

$$\|S_f\|_G := \sup_{z \in G} |S_f| \eta_G(z)^{-2},$$

where  $\eta_G$  is a Poincaré density of  $G$ . One of the important properties of  $\|S_f\|$  is the following; Let  $f$  be meromorphic on  $G$  and  $g$  and  $h$  Möbius transformations, then  $\|S_f\|_G = \|S_{h \circ f \circ g}\|_{g^{-1}(G)}$ . It shows that  $\|S_f\|$  is completely invariant under compositions of Möbius transformations. We note that if  $G = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ , then  $\|S_f\|_{\mathbb{D}} = (1 - |z|)^2 |S_f|$ . For later use, we denote  $\|S_f\|_{\mathbb{D}}$  by simply  $\|S_f\|$ .

**1.3. Bers embedding of Teichmüller spaces.** Let us consider the family  $\mathcal{F}$  of all quasiconformal automorphisms of the upper half-plane  $\mathbb{H}^+$ . Since all mappings in  $\mathcal{F}$  can be extended to homeomorphic self-mappings of the closure of  $\mathbb{H}^+$ , all components of  $\mathcal{F}$  are recognized as self-homeomorphisms of  $\overline{\mathbb{H}^+}$ . We define an equivalence relation  $\sim$  on  $\mathcal{F}$  according to which  $f \sim g$  for all  $f, g \in \mathcal{F}$  if and only if there exists a holomorphic automorphism  $M$  of  $\mathbb{H}^+$ , a Möbius transformation having the form  $M(z) = (az + b)/(cz + d)$ ,  $a, b, c, d \in \mathbb{R}$ , such that  $f \circ M = g$  on  $\mathbb{R}$ . The equivalence relation on  $\mathcal{F}$  forms the quotient space  $\mathcal{F}/\sim$ , which is called the **universal Teichmüller space** and denoted by  $\mathcal{T}$ . Theorem 1.1 with (1.3) tells us that there is a one-to-one correspondence between  $\mathcal{F}$  and  $B(\mathbb{H}^+)$ . If  $f \sim g$ , then the corresponding complex dilatations  $\mu_f$  and  $\mu_g$  are also said to be **equivalent**.

Another equivalent class of  $\mathcal{T}$  is given by the following profound observation due to Bers [Ber60]. Let  $\mu \in B(\mathbb{H}^+)$ . We extend  $\mu$  to the lower half-plane  $\mathbb{H}^- := \{z \in \mathbb{C} : \text{Im } z < 0\}$

by putting 0 everywhere. By Theorem 1.1, there exists a quasiconformal mapping  $f^\mu$  fixing 0, 1,  $\infty$  associated with such an extended  $\mu$ . Then  $f^\mu|_{\mathbb{H}^-}$  is conformal.

**Theorem 1.2** (see e.g. [Leh87, Theorem III-1.2]). *The complex dilatations  $\mu$  and  $\nu$  are equivalent if and only if  $f^\mu|_{\mathbb{H}^-} \equiv f^\nu|_{\mathbb{H}^-}$ .*

By the above theorem, the universal Teichmüller space  $\mathcal{T}$  can be understood as the set of the normalized conformal mappings  $f^\mu|_{\mathbb{H}^-}$  which can be extended quasiconformally to the upper half-plane  $\mathbb{H}^+$ . Recall that for a Möbius transformation  $f$  we have  $S_{f \circ g} = S_g$ . Therefore, it is natural to consider the mapping

$$\mathcal{T} \ni [f] \mapsto S_{f^\mu|_{\mathbb{H}^-}} \in \mathcal{Q}, \quad (1.4)$$

between  $\mathcal{T}$  and  $\mathcal{Q}$ , where  $\mathcal{Q}$  is a space of functions  $\phi$  holomorphic in  $\mathbb{H}^-$  for which the hyperbolic sup norm  $\|\phi\| := \sup_{z \in \mathbb{H}^-} (\operatorname{Im} z)^2 |\phi(z)|$  is finite.

In order to investigate a detailed property of the mapping (1.4), we define a metric on  $\mathcal{T}$  by

$$d_t(p, q) := \frac{1}{2} \inf_{p, q \in \mathcal{T}} \{\log K(g \circ f^{-1}) : f \in p, g \in q\},$$

where  $K(f) := (1 + k(f))/(1 - k(f))$ , and on  $\mathcal{Q}$  by

$$d_q(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|_\infty.$$

$d_t$  is called the **Teichmüller distance**. As a consequence of the fact that  $d_t$  and  $d_q$  are topologically equivalent, we obtain the following theorem which provides a new model of the universal Teichmüller space.

**Theorem 1.3.** *The mapping (1.4) is a homeomorphism of the universal Teichmüller space  $\mathcal{T}$  onto its image in  $\mathcal{Q}$ .*

The mapping (1.4) is called the **Bers embedding** of Teichmüller space. We denote the image of  $\mathcal{T}$  under (1.4) by  $\mathcal{T}_1$ . It is known that  $\mathcal{T}_1$  is a bounded, connected and open subset of  $\mathcal{Q}$  ([Ahl63]).

From the viewpoint of the theory of univalent functions,  $\mathcal{T}_1$  is characterized as follows. Let  $\mathcal{A}$  be the family of functions  $f$  holomorphic in  $\mathbb{D}$  with  $f(0) = 0$  and  $f'(0) = 1$  and  $\mathcal{S}$  be the subfamily of  $\mathcal{A}$  whose components are univalent on  $\mathbb{D}$ . We define  $\mathcal{S}(k)$  and  $\mathcal{S}^*(k)$  as the families of functions in  $\mathcal{S}$  which can be extended to  $k$ -quasiconformal mappings of  $\mathbb{C}$  and  $\widehat{\mathbb{C}}$ . Set  $\mathcal{S}(1) := \cup_{k \in [0,1)} \mathcal{S}(k)$ . Then  $\mathcal{T}_1$  is written by

$$\mathcal{T}_1 = \{S_f : f \in \mathcal{S}(1)\}.$$

We give a short account of the relation to the Teichmüller spaces. Let  $S_1$  and  $S_2$  be Riemann surfaces and  $G_1$  and  $G_2$  the covering groups of  $\mathbb{H}$  over  $S_1$  and  $S_2$ , respectively. For the Riemann surfaces  $S_1$  and  $S_2$ , the Teichmüller spaces  $\mathcal{T}_{S_1}$  and  $\mathcal{T}_{S_2}$  are defined. If  $G_1$  is a subgroup of  $G_2$ , then the relation  $\mathcal{T}_{S_2} \subset \mathcal{T}_{S_1}$  holds. In particular, if  $G_1$  is trivial, then  $\mathcal{T}_{S_1}$  is the universal Teichmüller space which includes all the other Teichmüller spaces as subspaces. For this reason the name “universal” is used to  $\mathcal{T}_1$ .

## 2. QUASICONFORMAL EXTENSIONS OF UNIVALENT FUNCTIONS

In Section 1 we have introduced  $\mathcal{S}(k)$  to characterize the universal Teichmüller space  $\mathcal{T}_1$ . Before entering the main part concerning with Loewner theory, we present some results of the general study of quasiconformal extensions.

**2.1. Examples of quasiconformal extensions.** For a given conformal mapping  $f$  of a domain  $D$ , we say that  $f$  has a **quasiconformal extension** to  $\mathbb{C}$  if there exists a  $k$ -quasiconformal mapping  $F$  such that its restriction  $F|_D$  is equal to  $f$ . For some fundamental conformal mappings, we can construct quasiconformal extensions explicitly. Below we summarize such examples which are sometimes useful. Some more examples can be found in [IT92, p.78]. We remark that (1.1) is written by the polar coordinates as

$$\left| \frac{ir\partial_r f(re^{i\theta}) - \partial_\theta f(re^{i\theta})}{ir\partial_r f(re^{i\theta}) + \partial_\theta f(re^{i\theta})} \right| \leq k,$$

where  $\partial_r := \partial/\partial r$  and  $\partial_\theta := \partial/\partial \theta$ .

**Example 2.1.** A very simple but important example is

$$f(z) = \begin{cases} z + \frac{k}{z}, & |z| > 1, \\ z + k\bar{z}, & |z| < 1. \end{cases}$$

where  $k \in [0, 1)$ . Then  $|f_{\bar{z}}/f_z| = k$ . The case  $k = 1$  reflects the Joukowski transform in  $|z| > 1$ , though in this case  $f$  is not a quasiconformal mapping any more.

**Example 2.2.** An identity mapping of  $\mathbb{D}$  has trivially a quasiconformal extension. In fact, the following extension,

$$f(z) = \begin{cases} re^{i\theta}, & r < 1, \\ \phi(r)e^{i\theta}, & r \geq 1, \end{cases}$$

is given, where  $\phi : [1, \infty) \rightarrow [1, \infty)$  is bi-Lipschitz continuous and injective with  $\phi(1) = 1$  and  $\phi(\infty) = \infty$ . The maximal dilatation is given by

$$|\mu_f| = \left| \frac{\phi(r) - r\phi'(r)}{\phi(r) + r\phi'(r)} \right|.$$

Let  $M > 1$  be a Lipschitz constant. Then  $1/M \leq \phi'(r) \leq M$  and  $1/M \leq \phi(r)/r \leq M$  and therefore  $1 \leq r\phi'(r)/\phi(r) \leq M^2$ . We conclude that the extension is  $|\mu_f| \leq |M^2 - 1|/|M^2 + 1|$ -quasiconformal.

**Example 2.3.** Let  $K(z) := (1+z)/(1-z)$  be a Cayley map and  $P_\beta(z) := z^\beta$ . For a fixed  $\beta \in (0, 2)$ , the function

$$f(z) := (P_\beta \circ K)(z)$$

maps  $\mathbb{D}$  onto the sector domain  $\Delta(-\beta, \beta) := \{z : -\pi\beta/2 < \arg z < \pi\beta/2\}$ . We shall construct a quasiconformal extension of  $f$ . The function  $g(z) := (-P_{2-\beta} \circ -K)(z)$  maps  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto  $\Delta(\beta, 4-\beta)$ . But in this case  $f(e^{i\theta}) \neq g(e^{i\theta})$  for each  $\theta \in (0, 2\pi)$ . In order to sew these two functions on their boundaries, define  $h(re^{i\theta}) := r^{\beta/(2-\beta)}e^{i\theta}$ . Then  $(-P_{2-\beta} \circ h \circ -K)(z)$  takes the same value as  $f$  on  $\partial\mathbb{D}$ . Hence it gives a quasiconformal extension of  $f$ . A calculation shows that its maximal dilatation is  $|1 - \beta|$ .

**Example 2.4.** For a given  $\lambda \in (-\pi/2, \pi/2)$ , a function defined by

$$f(re^{i\theta}) = e^{i\theta} \exp(e^{i\lambda} \log r)$$

is a  $\tan(\lambda/2)$ -quasiconformal mapping of  $\mathbb{C}$  onto  $\mathbb{C}$ . On the other hand, since the above  $f$  maps a radial segment  $[0, \infty)$  to a logarithmic spiral, it is not differentiable at the origin. By calculation we have  $|f| = \exp(\cos \lambda \log r)$  and  $\arg f = \theta + \sin \lambda \log r$ . Therefore  $f$  with a proper rotation gives a  $\tan(\lambda/2)$ -quasiconformal extension for a function  $f(z) = cz$  on  $\mathbb{D}$  or  $\mathbb{D}^* := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , where  $c$  is some constant.

**Example 2.5.** The followings are typical functions in  $\mathcal{S}$  which do not have any quasiconformal extensions,

$$f(z) = \frac{z}{(1-z)^2} \quad \text{and} \quad f(z) = z - \frac{z^2}{2}.$$

The first one is known as the Koebe function which maps  $\mathbb{D}$  onto  $\mathbb{C} \setminus (-\infty, -1/4]$ . There does not exist a homeomorphism which maps  $\mathbb{D}^*$  onto  $(-\infty, -1/4]$ . As for the second function,  $\partial\mathbb{D}$  is mapped to a cardioid which has a cusp at  $z = 1$ .

**2.2. Extremal problems on  $\mathcal{S}(k)$ .** In order to investigate the structure of the family of functions, the extremal problems sometimes provide us quite beneficial information. One of the most known problems is the Bieberbach conjecture [Bie16], solved by de Branges [dB85], which asks exact estimates of coefficients of the Taylor expansions of the functions in the class  $\mathcal{S}$ . A similar problem for  $\mathcal{S}(k)$  and  $\Sigma(k)$  (defined below) were proposed, and many mathematicians have worked on this problem. We note that in spite of such a circumstance, there are many open problems in this field including the coefficient problem.

Let  $\Sigma$  be the class of univalent holomorphic maps  $g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^{-n}$  maps  $\mathbb{D}^* := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  into  $\widehat{\mathbb{C}} \setminus \{0\}$ . If  $f \in \Sigma$  has a  $k$ -quasiconformal extension to  $\widehat{\mathbb{C}}$ , then we say that  $f$  belongs to the class  $\Sigma(k)$ .

Our argument is done on the following fact.

**Theorem 2.6.**  $\mathcal{S}(k)$ ,  $\mathcal{S}^*(k)$ ,  $\Sigma(k)$  are compact families.

Kühnau gave a fundamental contribution to the coefficient problem with the variational method.

**Theorem 2.7** ([Küh69]). Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(k)$  and  $g(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n} \in \Sigma(k)$ . Then the followings hold;  $|b_0| \leq 2k$ ,  $|b_1| \leq k$  and  $|a_3 - a_2^2| \leq k$ , in particular  $|a_2| \leq 2k$ .

We note that in the case when  $k = 1$  we obtain estimates for the class  $\mathcal{S}$  and  $\Sigma$ .

As more general approach to this problem, the distortion theorem for bounded functional was studied. We basically follow the description of the survey paper by Krushkal [Kru05b, Chapter 3]. The reader is also referred to [KK83].

Let  $E \subset \widehat{\mathbb{C}}$  be a measurable set whose complement  $E^* := \widehat{\mathbb{C}} \setminus E$  has positive measure, and set

$$B^*(E) := \{\mu \in B(\widehat{\mathbb{C}}) : \mu|_E \in B(E) \text{ and } \mu|_{E^*} = 0\}.$$

Denote by  $Q(E)$  a family of normalized quasiconformal mappings  $f_\mu : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  where  $\mu \in B(E)$ , and  $Q_k(E) := \{f \in Q(E) : \|\mu_f\| \leq k\}$  for a  $k \in [0, 1)$ . Now let  $F : Q(E) \rightarrow \widehat{\mathbb{C}}$  be a non-trivial holomorphic functional, where holomorphic means that it is complex Gateaux differentiable. Lastly, set  $\|F\|_1 := \max_{f \in Q(E)} |F(f)|$  and  $\|F\|_k := \max_{f \in Q_k(E)} |F(f)|$ .

**Theorem 2.8.** Let  $F(Q(E))$  be bounded. Then we have  $\|F\|_k \leq k\|F\|_1$ .

Some applications of the theorem are demonstrated in [Kru05b, Chapter 3.4]. One of them is the distortion theorem for the class  $\mathcal{S}(k)$  (see also [Gut73, Corollary 7]);

$$\left( \frac{1-z}{1+z} \right)^k \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \left( \frac{1+z}{1-z} \right)^k.$$

For more results and their proofs, see [Sch75], [Kru05b], [Kru05a].

The estimate of  $|a_2|$  for the class  $\mathcal{S}^*(k)$  is obtained by Schiffer and Schober.

**Theorem 2.9** ([SS76]). *For all  $f \in \mathcal{S}^*(k)$ , we have  $|a_2| \leq 2 - 4\pi^{-2}(\arccos k)^{-2}$ .*

Since the class  $\mathcal{S}^*(k)$  is closed with respect to the Koebe transform

$$f_K(z) := \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + \left( \frac{1}{2}(1-|\zeta|^2) \frac{f''(\zeta)}{f'(\zeta)} - \bar{\zeta} \right) z^2 + \cdots, \quad (2.1)$$

we have the fundamental estimate for  $\mathcal{S}^*(k)$

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{(4 - 8\pi^{-2}(\arccos k)^{-2})|z|}{1-|z|^2}.$$

Following the standard argument for the class  $\mathcal{S}$  (see e.g. [Pom75, pp.21-22]), we have distortions of  $f$  and  $f'$  for  $\mathcal{S}^*(k)$ . We note that the same method as this is not valid for the class  $\mathcal{S}(k)$  because the Koebe transform (2.1) does not fix  $\infty$  except the case  $\zeta = 0$ .

As is written before, while the coefficient problem has been completely solved in the class  $\mathcal{S}$ , the question remains open for the class  $\mathcal{S}(k)$ . However, if we restrict ourselves to that  $k$  is sufficiently small, then the complete result is established by Krushkal.

**Theorem 2.10** ([Kru88, Kru95]). *For a function  $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}(k)$ , we have the sharp estimate*

$$|a_n| \leq \frac{2k}{n-1}. \quad (2.2)$$

*This estimate is valid only if  $k \leq 1/(n^2 + 1)$ .*

The extremal function of the estimate (2.2) is given by

$$f_2(z) := \frac{z}{(1-kz)^2} \quad (k \in [0, 1)),$$

$$f_n(z) := (f_2(z^{n-1}))^{1/(n-1)} = z + \frac{2k}{n-1}z^n + \cdots \quad n = 3, 4, \dots$$

To see that  $f_n \in \mathcal{S}(k)$ , calculate  $zf'_n(z)/f_n(z)$  and apply the quasiconformal extension criterion for starlike functions in Section 3.4.

**2.3. Sufficient conditions for  $\mathcal{S}(k)$ .** Since Bers introduced a new model of the universal Teichmüller space, numerous sufficient conditions for the class  $\mathcal{S}(k)$  have been obtained. In this subsection we introduce only a few remarkable results.

In 1962, the first sufficient condition for  $\mathcal{S}(k)$  was provided by Ahlfors and Weil.

**Theorem 2.11** ([AW62]). *Let  $f$  be a non-constant meromorphic function defined on  $\mathbb{D}$  and  $k \in [0, 1)$  be a constant. If  $f$  satisfies  $\|S_f\| \leq k$ , then  $f$  can be extended to a quasiconformal mapping  $F$  to  $\widehat{\mathbb{C}}$ . In this case the dilatation  $\mu_F$  is given by*

$$\mu_F(z) := \begin{cases} -\frac{1}{2}(|z|^2 - 1)^2 S_f\left(\frac{1}{\bar{z}}\right) \frac{1}{\bar{z}^4}, & |z| > 1 \\ 0, & |z| < 1. \end{cases}$$

1972, Becker showed the sufficient condition in connection with the pre-schwarzian derivative. Later it was generalized by Ahlfors.

**Theorem 2.12** ([Ahl74]). *Let  $f \in \mathcal{A}$ . If there exists a  $k \in [0, 1)$  such that for a constant  $c \in \mathbb{C}$  satisfying*

$$\left| c|z|^2 + (1-|z|^2) \frac{f''(z)}{f'(z)} \right| \leq k \quad (2.3)$$

*for all  $z \in \mathbb{D}$ , then  $f \in \mathcal{S}(k)$ .*

The case when  $c = 0$  is due to Becker [Bec72]. Remark that the condition  $|c| \leq k$  which was stated in the original form is embedded in the inequality (2.3) (see [Hot10]).

It is known that many univalence criteria are refined to quasiconformal extension criteria. For instance, Fait, Krzyż and Zygmunt proved the following theorem which is the refinement of the definition of strongly starlike functions (for its definition, see Section 3.3).

**Theorem 2.13** ([FKZ76]). *Every strongly starlike functions of order  $\alpha$  has a  $\sin(\pi\alpha/2)$ -quasiconformal extension to  $\mathbb{C}$ .*

This is generalized to strongly spiral-like functions [Sug12]. Some more results are obtained in [Bro84, Hot09] with explicit quasiconformal extensions which correspond to each subclass of  $\mathcal{S}$ . In particular, in [Hot09] the research relies on the (classical) Loewner theory, which will be mentioned in the next section.

Sugawa approached this problem by means of the holomorphic motions with extended  $\lambda$ -Lemma ([MSS83], [Slo91]).

**Theorem 2.14** ([Sug99]). *Let  $k \in [0, 1)$  be a constant. For a given  $f \in \mathcal{A}$ , let  $p$  denote one of the quantities  $zf'(z)/f(z)$ ,  $1 + zf''(z)/f'(z)$  and  $f'(z)$ . If  $|(1 - p(z))/(1 + p(z))| \leq k$  for all  $z \in \mathbb{D}$ , then  $f \in \mathcal{S}(k)$ .*

We note that in most of the sufficient conditions of quasiconformal extensions including the above theorems the case  $k = 1$  reflects a univalence criterion.

### 3. CLASSICAL LOEWNER THEORY

The idea of the parametric representation method of conformal maps was introduced by Löwner [Löw23] and later developed by Kufarev [Kuf43] and Pommerenke [Pom65]. It describes a time-parametrized conformal map on  $\mathbb{D}$  whose image is a continuously increasing simply-connected domain. The key point is that such a family can be represented by a partial differential equation. Loewner's approach also made a significant contribution to quasiconformal extensions of univalent functions. This method was discovered by Becker.

Since our focus in this note is univalent functions with quasiconformal extensions, we will deal with Loewner chains in the sense of Pommerenke. For one-slit maps as Löwner originally considered, see e.g. [dMG] which also contains a list of references. The reader is also referred to [Hen86, Chapter 19] and [GK03, Chapter 3].

**3.1. Classical Loewner chains.** Let  $f_t(z) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$  be a function defined on  $\mathbb{D} \times [0, \infty)$ .  $f_t$  is said to be a (**classical**) **Loewner chain** if  $f_t$  satisfies the conditions;

1.  $f_t$  is holomorphic and univalent in  $\mathbb{D}$  for each  $t \in [0, \infty)$ ,
2.  $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$  for all  $0 \leq s < t < \infty$ .

We note that in this definition  $f_t$  is not assumed to be differentiable by  $t$ . One can also characterize it geometrically. Let  $\{D_t\}_{t \geq 0}$  be a family of simply-connected domains which has the following properties;

- 1'.  $0 \in D_0$ ,
- 2'.  $D_s \subsetneq D_t$  for all  $0 \leq s < t < \infty$ ,
- 3'.  $D_{t_n} \rightarrow D_t$  if  $t_n \rightarrow t < \infty$  and  $D_{t_n} \rightarrow \mathbb{C}$  if  $t_n \rightarrow \infty$  ( $n \rightarrow \infty$ ), in the sense of the kernel convergence.

Then by the Riemann mapping theorem there exists a family of conformal mappings  $\{f_t\}_{t \geq 0}$  such that  $f_t(0) = 0$  and  $f'_t(0) > 0$  for all  $t \geq 0$ . We note that  $f'_t(0)$  is strictly

increasing with respect to  $t \geq 0$ , for otherwise  $f_t^{-1} \circ f_s$  is an identity which contradicts  $D_s \subsetneq D_t$ . After rescaling as  $f_0 \in \mathcal{S}$  and reparametrizing as  $f'_t(0) = e^t$ , we obtain a Loewner chain.

The following necessary and sufficient condition for a Loewner chain is known by Pommerenke.

**Theorem 3.1** ([Pom65, Pom75]). *Let  $0 < r_0 \leq 1$ . Let  $f_t(z) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$  be a function defined on  $\mathbb{D} \times [0, \infty)$ . Then  $f_t$  is a Loewner chain if and only if the following two conditions are satisfied;*

- (i)  *$f_t$  is holomorphic in  $z \in \mathbb{D}_{r_0}$  for each  $t \in [0, \infty)$ , absolutely continuous in  $t \in [0, \infty)$  for each  $z \in \mathbb{D}_{r_0}$  and satisfies*

$$|f_t| \leq K_0 e^t \quad (z \in \mathbb{D}_{r_0}, t \in [0, \infty))$$

*for some positive constant  $K_0$ .*

- (ii) *There exists a function  $p(z, t)$  analytic in  $z \in \mathbb{D}$  for each  $t \in [0, \infty)$  and measurable in  $t \in [0, \infty)$  for each  $z \in \mathbb{D}$  satisfying*

$$\operatorname{Re} p(z, t) > 0 \quad (z \in \mathbb{D}, t \in [0, \infty))$$

*such that*

$$\dot{f}_t(z) = z f'_t(z) p(z, t) \quad (z \in \mathbb{D}_{r_0}, \text{ a.e. } t \in [0, \infty)) \quad (3.1)$$

*where  $\dot{f} = \partial f / \partial t$  and  $f' = \partial f / \partial z$ .*

The partial differential equation (3.1) is called the **Loewner-Kufarev PDE**, and the function  $p$  in (3.1) is called a **Herglotz function**. We observe that (3.1) describes an expanding flow of the image domain  $f_t(\mathbb{D})$  of a Loewner chain. Indeed, (3.1) can be written as

$$|\arg \dot{f}_t(z) - \arg z f'_t(z)| = |\arg p(z, t)| < \frac{\pi}{2}.$$

It implies that the velocity vector  $\dot{f}_t$  at a boundary point of the domain  $f_t(\mathbb{D}_r)$  points out of this set and therefore all points on  $\partial f_t(\mathbb{D}_r)$  moves to outside of  $\overline{f_t(\mathbb{D}_r)}$  when  $t$  increases. The next property is also important.

**Theorem 3.2.** *For any  $f \in \mathcal{S}$ , there exists a Loewner chain  $f_t$  such that  $f_0 = f$ .*

For a Loewner chain  $f_t$ , the function  $\varphi_{s,t}(z) := f_t^{-1} \circ f_s(z)$  ( $0 \leq s \leq t < \infty$ ), called an **evolution family**, plays a core role in Loewner theory. By definition  $\varphi_{s,t}$  is a univalent self-map of  $\mathbb{D}$  with  $\varphi_{s,t}(0) = 0$  and  $\varphi'_{s,t}(0) = e^{s-t}$  and has the semigroup property  $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$  for all  $0 \leq s \leq u \leq t < \infty$ . Since  $f_t(\varphi_{s,t}(z)) = f_s$ , differentiating both sides of the equation with respect to  $t$ , we have  $\dot{f}_t(\varphi_{s,t}) + f'_t(\varphi_{s,t}) \dot{\varphi}_{s,t} = 0$  and hence one can obtain by (3.1)

$$\dot{\varphi}_{s,t}(z) = \varphi_{s,t}(z) p(\varphi_{s,t}(z), t). \quad (3.2)$$

This is called the **Loewner-Kufarev ODE**. The following is the basic result about existence and uniqueness of a solution of the ODE.

**Theorem 3.3.** *Suppose that a function  $p(z, t)$  is holomorphic in  $z \in \mathbb{D}$  and measurable in  $t \in [0, \infty)$  satisfying  $\operatorname{Re} p(z, t) > 0$  for all  $z \in \mathbb{D}$  and  $t \in [0, \infty)$ . Then, for each fixed  $z \in \mathbb{D}$  and  $s \in [0, \infty)$ , the initial value problem*

$$\frac{dw}{dt} = -w p(w, t)$$



for almost all  $t \in [s, \infty)$  has a unique absolutely continuous solution  $w(t)$  with the initial condition  $w(s) = z$ . If we write  $\varphi_{s,t}(z) := w(t)$ , then  $\varphi_{s,t}$  is an evolution family. Further, the function  $f_s(z)$  defined by

$$f_s(z) := \lim_{t \rightarrow \infty} e^t \varphi_{s,t}(z) \quad (3.3)$$

exists locally uniformly in  $z \in \mathbb{D}$  and is a Loewner chain.

Conversely, if  $f_t$  is a Loewner chain and  $\varphi_{s,t}$  is an evolution family associated with  $f_t$ . Then for almost all fixed  $t \in [s, \infty)$ ,  $\varphi_{s,t}$  satisfies

$$\dot{\varphi}_{s,t} = -\varphi_{s,t} p(\varphi_{s,t}, t)$$

for all  $z \in \mathbb{D}$ , and (3.3) is satisfied.

In the first assertion of Theorem 3.3, it may happen that two different Herglotz functions  $p_1$  and  $p_2$  generate the same evolution family  $\varphi_{s,t}$ . Then  $p_1(z, t) = p_2(z, t)$  for almost all  $t \geq 0$ . Hence Theorem 3.3 claims that there is a one-to-one correspondence between an evolution family and a Herglotz function in such a sense.

**3.2. Becker's theorem.** An interesting method connecting Loewner theory and quasiconformal extensions was obtained by Becker.

**Theorem 3.4** ([Bec72], [Bec80]). *Suppose that  $f_t$  is a Loewner chain for which  $p(z, t)$  in (3.1) satisfying the condition*

$$p(z, t) \in U(k) := \left\{ w \in \mathbb{C} : \left| \frac{1-w}{1+w} \right| \leq k \right\} \quad (3.4)$$

i.e.,  $p(z, t)$  lies in the closed hyperbolic disk  $U(k)$  in the right half plane centered at 1 with radius  $\operatorname{arctanh} k$ , for all  $z \in \mathbb{D}$  and almost all  $t \geq 0$ . Then  $f_t$  admits a continuous extension to  $\overline{\mathbb{D}}$  for each  $t \geq 0$  and the map  $F$  defined by

$$F(re^{i\theta}) = \begin{cases} f_0(re^{i\theta}), & \text{if } r < 1, \\ f_{\log r}(e^{i\theta}), & \text{if } r \geq 1, \end{cases} \quad (3.5)$$

is a  $k$ -quasiconformal extension of  $f_0$  to  $\mathbb{C}$ .

The idea of the theorem is the following. By Koebe's 1/4-Theorem,  $f_t(\mathbb{D})$  must contain the disk whose center is 0 with radius  $e^t/4$ , and hence  $f_t(\mathbb{D})$  tends to  $\mathbb{C}$  as  $t \rightarrow \infty$ . This fact implies that the boundary  $\partial f_t(\mathbb{D})$  runs throughout on  $\mathbb{C} \setminus f_0(\mathbb{D})$ . Therefore the mapping  $F : \mathbb{D}^* \rightarrow \mathbb{C} \setminus f_0(\mathbb{D})$  is constructed by (3.5) which gives a correspondence between the circle  $\{|z| = e^t\}$  and the boundary  $\partial f_t(\mathbb{D})$ . Its quasiconformality follows from the condition (3.4).

Becker generalized Theorem 3.4 by introducing an inverse version of Loewner chains. Let  $\omega_t(z) = \sum_{n=1}^{\infty} b_n(t)z^n$ ,  $b_1(t) \neq 0$ , be a function defined on  $\mathbb{D} \times [0, \infty)$ , where  $b_1(t)$  is a complex-valued, locally absolutely continuous function on  $[0, \infty)$ . Then  $\omega_t$  is said to be an **inverse Loewner chain** if

1.  $\omega_t$  is univalent in  $\mathbb{D}$  for each  $t \geq 0$ ,
2.  $|b_1(t)|$  decreases strictly monotonically as  $t$  increases, and  $\lim_{t \rightarrow \infty} |b_1(t)| \rightarrow 0$ .
3.  $\omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D})$  for  $0 \leq s < t < \infty$ ,
4.  $\omega_0(z) = z$  and  $\omega_s(0) = \omega_t(0)$  for  $0 \leq s \leq t < \infty$ .

$\omega$  also satisfies the partial differential equation

$$\dot{\omega}_t(z) = -z\omega'_t(z)q(z, t) \quad (z \in \mathbb{D}, \text{ a.e. } t \geq 0), \quad (3.6)$$

where  $q$  is a Herglotz function. Conversely, we can construct an inverse Loewner chain by means of (3.6) according to the following lemma:

**Lemma 3.5.** *Let  $q(z, t)$  be a Herglotz function. Suppose that  $q(0, t)$  be locally integrable in  $[0, \infty)$  with  $\int_0^\infty \operatorname{Re} q(0, t) dt = \infty$ . Then there exists an inverse Loewner chain  $\omega_t$  with (3.6).*

By applying the notion of an inverse Loewner chain, we obtain a generalization of Becker's result.

**Theorem 3.6** ([Bet92]). *Let  $k \in [0, 1)$ . Let  $f_t$  be a Loewner chain for which  $p(z, t)$  in (3.1) satisfying the condition*

$$\left| \frac{p(z, t) - \overline{q(z, t)}}{p(z, t) + q(z, t)} \right| \leq k \quad (z \in \mathbb{D}, \text{ a.e. } t \geq 0),$$

where  $q(z, t)$  is a Herglotz function. Let  $\omega_t$  be the inverse Loewner chain which is generated with  $q$  by Lemma 3.5. Then  $f_t$  and  $\omega_t$  are continuous and injective on  $\overline{\mathbb{D}}$  for each  $t \geq 0$ , and  $f_0$  has a  $k$ -quasiconformal extension  $F : \mathbb{C} \rightarrow \mathbb{C}$  which is defined by

$$F\left(\frac{1}{\omega_t(e^{i\theta})}\right) = f_t(e^{i\theta}) \quad (\theta \in [0, 2\pi), t \geq 0).$$

We obtain Becker's result for  $q(z, t) = 1$ . In this case an inverse Loewner chain is given by  $\omega_t(z) = e^{-t}z$ . Further, choosing  $\omega$  as  $p = q$ , we have the following corollary:

**Corollary 3.7** ([Bet92]). *Let  $\alpha \in [0, 1)$ . Suppose that  $f_t$  is a Loewner chain for which  $p(z, t)$  in (3.1) satisfies*

$$p(z, t) \in \Delta(-\alpha, \alpha) = \{z : -\alpha\pi/2 \leq \arg z \leq \alpha\pi/2\}$$

for all  $z \in \mathbb{D}$  and almost all  $t \in [0, \infty)$ . Then  $f_t$  admits a continuous extension to  $\overline{\mathbb{D}}$  for each  $t \geq 0$  and  $f_0$  has a  $\sin \alpha\pi/2$ -quasiconformal extension to  $\mathbb{C}$ .

Corollary 3.7 does not include Theorem 3.4 in view of the dilatation of the extended quasiconformal map. In fact, the following relation holds;

$$U(k) \subset \Delta(-k_0, k_0) \quad \text{where} \quad k_0 := \frac{2}{\pi} \arcsin\left(\frac{2k}{1-k^2}\right) \geq k.$$

Remark that  $k_0 = k$  if and only if  $k = 0$ .

In contrast to Becker's quasiconformal extension theorem, the theorem due to Betker does not always give a quasiconformal extension explicitly. The reason is based on the fact that in general it is difficult to express an inverse Loewner chain  $\omega_t$  which has the same Herglotz function as a given Loewner chain  $f_t$  in an explicit form. For details, see [HW, Section 5]

**3.3. Applications to the theory of univalent functions.** Here we will see some applications of Theorem 3.1 and Theorem 3.4. In order to find Loewner chains which corresponds to the typical subclasses of  $\mathcal{S}$ , we need to observe their geometric features. Some Loewner chains are not normalized as  $f'(0) = e^t$ . In [Hot11], it is discussed that Theorem 3.1 and Theorem 3.4 work well without such a normalization. In fact, a Loewner chain is generalized for a function  $f_t(z) = \sum_{n=1}^\infty a_n(t)z^n$  where  $a_1(t) \neq 0$  is a complex-valued, locally absolutely continuous function on  $[0, \infty)$  with  $\lim_{n \rightarrow \infty} |a_1(t)| = \infty$ . Further, either

condition that  $|a_1(t)|$  is strictly increase with respect to  $t \in [0, \infty)$  or  $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$  for all  $0 \leq s < t < \infty$  should be assumed.

**I. Convex functions.** A function  $f \in \mathcal{S}$  is said to be **convex** and belongs to  $\mathcal{K}$  if  $f(\mathbb{D})$  is a convex domain. It is known that  $f \in \mathcal{K}$  if and only if  $\operatorname{Re}[1 + (zf''(z)/f'(z))] > 0$  for all  $z \in \mathbb{D}$ . A flow of the expansion for a convex function is considered as following. If a boundary point  $\zeta \in \partial f(\mathbb{D})$  moves to the direction of their normal vector  $\zeta f'(\zeta)$  according to the parameter  $t$  increases, then  $\zeta$  always runs on the exterior of  $f(\mathbb{D})$  and their trajectories do not cross each other. In view of this, it is natural to set a Loewner chain as

$$f_t(z) = f(z) + t \cdot zf'(z) \quad (3.7)$$

Then we have  $1/p(z, t) = 1 + t \cdot [1 + (zf''(z)/f'(z))]$  and hence  $f_t$  is a Loewner chain if  $f \in \mathcal{K}$ .

**II. Starlike functions.** Next, consider a **starlike function** (with respect to 0), i.e., a function  $f \in \mathcal{S}$  such that for every  $z \in \mathbb{D}$  a segment which connects  $f(z)$  and 0 lies in  $f(\mathbb{D})$ . Denote by  $\mathcal{S}^*$  the family of starlike functions. An analytic characterization for starlike functions is  $\operatorname{Re} zf'(z)/f(z) > 0$  for all  $z \in \mathbb{D}$ . It follows from the definition that for a boundary point  $\zeta \in \partial f(\mathbb{D})$ , a ray  $\{tf(\zeta) : t \geq 1\}$  always lies in the exterior of  $f(\mathbb{D})$  and the rays are disjoint mutually. Hence a Loewner chain for  $\mathcal{S}^*$  is

$$f_t(z) := e^t f(z). \quad (3.8)$$

Then  $1/p(z, t) = zf'(z)/f(z)$  and therefore  $f_t$  is a Loewner chain if  $f \in \mathcal{S}^*$ . In the case of **spiral-like functions**, i.e., functions  $f \in \mathcal{S}$  defined by the condition  $\operatorname{Re} e^{-i\lambda} zf'(z)/f(z) > 0$  for some  $\lambda \in (-\pi/2, \pi/2)$ , a Loewner chain is given by

$$f_t(z) := e^{ct} f(z) \quad (3.9)$$

with  $c := e^{i\lambda}$  whose trajectories draw logarithmic spirals. The case  $\lambda = 0$  corresponds to starlike functions.

**III. Close-to-convex functions.** For a given  $f \in \mathcal{S}$ , if there exists a  $g \in \mathcal{S}^*$  such that  $\operatorname{Re} zf'(z)/g(z) > 0$  for all  $z \in \mathbb{D}$ , then  $f$  is said to be **close-to-convex** and we denote by  $f \in \mathcal{C}$ . The image  $f(\mathbb{D})$  by a close-to-convex function is known to be a **linearly accessible domain**, namely,  $\mathbb{C} \setminus f(\mathbb{D})$  is the union of closed half-lines which are mutually disjoint except their end points. We say that  $f$  is **linearly accessible** if  $f(\mathbb{D})$  is a linearly accessible domain.

A Loewner chain corresponding to the class  $\mathcal{C}$  is given by

$$f_t(z) := f(z) + t \cdot g(z). \quad (3.10)$$

Then  $1/p(z, t) = (zf'(z)/g(z)) + (zg'(z)/g(z))$  and hence  $\operatorname{Re} p(z, t) > 0$  for all  $z \in \mathbb{D}$  and  $t \geq 0$ . The validity of the chain (3.10) is given by the following consideration.

Let us take a fixed  $\rho \in (0, 1)$  and set  $f_\rho(z) := f(\rho z)/\rho$  and  $g_\rho(z) := g(\rho z)/\rho$ . Then  $f_t^\rho := f_\rho + tg_\rho$  is well-defined on  $\overline{\mathbb{D}}$ . For each boundary point  $\zeta_0 \in \partial \mathbb{D}$ ,

$$\gamma_{\zeta_0} := \{f_t^\rho(\zeta_0) : t \in [0, \infty)\}$$

define a half-line with an inclination of  $\arg g_\rho(\zeta_0)$ . Let  $\zeta_1 \in \partial \mathbb{D}$  be another boundary point with  $\zeta_1 \neq \zeta_0$ . Then  $\arg g_\rho(\zeta_0) \leq \arg g_\rho(\zeta_1)$ . Hence  $\gamma_{\zeta_0}$  and  $\gamma_{\zeta_1}$  do not have any intersection. Since  $f_t^\rho$  is a Loewner chain, in particular by the property  $f_t^\rho(\mathbb{D}) \rightarrow \mathbb{C}$  as  $t \rightarrow \infty$ ,  $\gamma_\zeta$  runs throughout  $\mathbb{C} \setminus f_\rho(\mathbb{D})$  if  $\zeta$  is taken from 0 to  $2\pi$ . Therefore  $\bigcup_{\zeta \in \partial \mathbb{D}} \gamma_\zeta = \mathbb{C} \setminus f_\rho(\mathbb{D})$  which proves that every  $f_\rho \in \mathcal{C}$  is linearly accessible. It is known that a family of linearly

accessible functions  $f \in \mathcal{S}$  is compact in the topology of locally uniform convergence ([Bie36]). Hence we conclude that  $f = \lim_{\rho \rightarrow 1} f_0^\rho \in \mathcal{C}$  is linearly accessible.

One can prove it without compactness of a family of linearly accessible functions. Let  $p_\zeta[f]$  be the prime end defined on a domain  $f(\mathbb{D})$  which corresponds to a boundary point  $\zeta \in \partial\mathbb{D}$  and  $I_\zeta[f]$  be the impression of the prime end  $p_\zeta[f]$ . It is known that there is a one-to-one correspondence among  $\zeta$ ,  $p_\zeta[f]$  and  $I_\zeta[f]$  (see [Pom92, Chapter 2]). Since  $g$  is starlike, for all  $w_g \in I_{\zeta_0}[g] \setminus \{\infty\}$ ,  $\arg w_g$  reflects one real value. Then redefine  $\gamma_{\zeta_0}$  as a family of half-lines (may consist of only one half-line) by

$$\gamma_{\zeta_0} := \{w_f + t \exp(i \arg w_g) : w_f \in I_{\zeta_0}[f] \setminus \{\infty\}, w_g \in I_{\zeta_0}[g] \setminus \{\infty\}, t \in [0, \infty)\}.$$

Then  $\bigcup_{\zeta \in \partial\mathbb{D}} \gamma_\zeta = \mathbb{C} \setminus f(\mathbb{D})$ , for otherwise there exists a point  $z \in \mathbb{C} \setminus f(\mathbb{D})$  such that  $z \notin \gamma_\zeta$  for any  $\zeta \in \partial\mathbb{D}$  which contradicts the fact that  $f_t$  is a Loewner chain. By choosing proper components of  $\bigcup_{\zeta \in \partial\mathbb{D}} \gamma_\zeta$ , a union of closed half-lines for that  $f(\mathbb{D})$  is a linearly accessible domain is given.

The Noshiro-Warschawski class is known as the special case of close-to-convex functions. Noshiro and Warschawski independently proved that if a function  $f \in \mathcal{A}$  satisfies  $\operatorname{Re} f'(z) > 0$  for all  $z \in \mathbb{D}$  then  $f \in \mathcal{S}$  (see e.g. [HW]). We denote a family of such functions by  $\mathcal{R}$ . It can be also shown by Loewner's method with the chain

$$f_t(z) := f(z) + t \cdot z. \quad (3.11)$$

By the above consideration, the following property can be derived.

**Proposition 3.8.** *For a function  $f \in \mathcal{R}$ , if  $f(\mathbb{D})$  is locally connected, then  $f(\mathbb{D})$  is Jordan.*

**IV. Bazilevič functions.** For real constants  $\alpha > 0$  and  $\beta \in \mathbb{R}$ , set  $\gamma = \alpha + i\beta$ . In 1955, Bazilevič [Baz55] showed that the function defined by

$$f(z) = \left[ (\alpha + i\beta) \int_0^z h(u) g(u)^\alpha u^{i\beta-1} du \right]^{1/(\alpha+i\beta)}$$

where  $g$  is a starlike univalent function and  $h$  is an analytic function with  $h(0) = 1$  satisfying  $\operatorname{Re}(e^{i\lambda} h) > 0$  in  $\mathbb{D}$  for some  $\lambda \in \mathbb{R}$  belongs to the class  $\mathcal{S}$ . It is called a **Bazilevič function of type  $(\alpha, \beta)$**  and we denote by  $\mathcal{B}(\alpha, \beta)$  the family of Bazilevič functions of type  $(\alpha, \beta)$ . A simple observation shows that  $f \in \mathcal{B}(\alpha, \beta)$  if and only if

$$\operatorname{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{g(z)} \right)^\alpha \left( \frac{f(z)}{z} \right)^\beta \right\} > 0 \quad (z \in \mathbb{D})$$

for some  $g \in \mathcal{S}^*$ . A Loewner chain for the class  $\mathcal{B}(\alpha, \beta)$  is known ([Pom65, p.166]) as

$$f_t(z) = (f(z)^\gamma + t \cdot \gamma g(z)^\alpha z^{i\beta})^{1/\gamma}. \quad (3.12)$$

By using the previous argument for close-to-convex functions, we can derive some geometric features for the class  $\mathcal{B}(\alpha, \beta)$ . We consider the simple case that  $f(\mathbb{D})$  and  $g(\mathbb{D})$  are locally connected. Then for each point  $\zeta_0 \in \partial\mathbb{D}$ , the curve  $\{\delta_{\zeta_0}(t) := (f(\zeta_0)^\gamma + t \cdot \gamma g(\zeta_0)^\alpha \zeta_0^{i\beta})^{1/\gamma} : t \in [0, \infty)\}$  is defined. Hence  $f(\mathbb{D})$  is described as a complement of a union of such curves.

Observe the behavior of the curve. If  $\beta > 0$  (or  $\beta < 0$ ), then it draws an asymptotically similar curve as a logarithmic spiral which evolves counterclockwise (or clockwise). On the other hand, in the case when  $\beta = 0$ , firstly it draws a spiral, then tends to a straight line as  $t$  gets large. In both cases, the curvature  $d_t \arg \delta'_{\zeta_0}(t) = \operatorname{Im} [\delta''_{\zeta_0}(t)/\delta'_{\zeta_0}(t)]$  is always positive or negative. From this fact one can construct functions which do not belong to

any  $\mathcal{B}(\alpha, \beta)$  easily. Consider a slit domain  $\mathbb{C} \setminus \gamma$ . If the curvature of the slit  $\gamma$  takes both positive and negative values (ex. similar as a sine curve), or  $\gamma$  is not smooth (ex. a similar curve as  $\{x \geq 0\} \cup \{iy : y \in (0, 1)\}$ ), then such slit domains cannot be images of  $\mathbb{D}$  under any  $f \in \mathcal{B}(\alpha, \beta)$ .

**3.4. Applications to quasiconformal extensions.** Applying Theorem 3.4 to the chains (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) we obtain quasiconformal extension criteria for each subclass of  $\mathcal{S}$  with explicit extensions. In this case the chains (3.7), (3.10), (3.11) and (3.12) should be reparametrized by  $e^t - 1$ . The theorems can be found in [Hot09, Hot11, HW11, Hot13]. Further, by Theorem 3.7 with the chains (3.8) and (3.9) we obtain quasiconformal extension criteria given by [FKZ76] and [Sug12]. For an explicit extension of these cases, see [HW].

The other typical example is Theorem 2.12, Ahlfors's quasiconformal extension criterion. It can be obtained by Theorem 3.4 with the chain

$$f_t(z) := f(e^{-t}z) + \frac{1}{1+c}(e^t - e^{-t})f'(e^{-t}z),$$

for then

$$\frac{1 - p(z, t)}{1 + p(z, t)} = \frac{zf'_t(z) - \dot{f}_t(z)}{zf'_t(z) + \dot{f}_t(z)} = c \frac{1}{e^{2t}} + \left(1 - \frac{1}{e^{2t}}\right) \frac{e^{-t}zf'(e^{-t}z)}{f''(e^{-t}z)}.$$

#### 4. MODERN LOEWNER THEORY

Recently a new approach to treat evolution families and Loewner chains in a quite general framework has been suggested by Bracci, Contreras, Díaz-Madrigal and Gumenyuk ([BCDM12], [BCDM09], [CDMG10]). It enables us to describe a variety of the dynamics of one-parameter family of conformal mappings. In this section we outline the theory of generalized evolution families and Loewner chains. The key fact is that there is an (essentially) one-to-one correspondence among evolution families, Herglotz vector fields and Loewner chains. We also present some new results about generalized Loewner chains with quasiconformal extensions.

**4.1. Semigroups of holomorphic mappings.** Let  $D$  be a simply connected domain in the complex plane  $\mathbb{C}$ . We denote the family of all holomorphic functions on  $D$  by  $\text{Hol}(D, \mathbb{C})$ . If  $f \in \text{Hol}(\mathbb{D}, \mathbb{C})$  is a self-mapping of  $\mathbb{D}$ , then we will denote the family of such functions by  $\text{Hol}(\mathbb{D})$ .

An easy consequence of the well-known Schwarz-Pick Lemma,  $f \in \text{Hol}(\mathbb{D}) \setminus \{\text{id}\}$  may have at most one fixed point in  $\mathbb{D}$ . If such a point exists, then it is called the **Denjoy-Wolff point** of  $f$ . On the other hand, if  $f$  does not have a fixed point in  $\mathbb{D}$ , then the Denjoy-Wolff theorem (see e.g. [ES10]) claims that there exists a unique boundary fixed point  $\angle \lim_{z \rightarrow \tau} f(z) = \tau \in \partial\mathbb{D}$  such that the sequence of iterates  $\{f^n\}_{n \in \mathbb{N}}$  converges to  $\tau$  locally uniformly, where  $\angle \lim$  denotes an angular (or non-tangential) limit, and  $f^n$  an  $n$ -th iterate of  $f$ , namely,  $f^1 := f$  and  $f^n := f^{n-1} \circ f$ . In this case the boundary point  $\tau$  is also called the **Denjoy-Wolff point**. A boundary fixed point is not always the Denjoy-Wolff point. A simple example is observed with a holomorphic automorphism of  $\mathbb{D}$ ,  $f(z) = (z + a)/(1 + \bar{a}z)$  with  $a \in \mathbb{D} \setminus \{0\}$ .  $f$  has two boundary fixed points  $\pm a/|a|$ , but only one  $a/|a|$  can be the Denjoy-Wolff point.

A family  $\{\phi_t\}_{t \geq 0}$  of holomorphic self-mappings of  $\mathbb{D}$  is called a **one-parameter semigroup** if

1.  $\phi_0 = id_{\mathbb{D}}$ ,
2.  $\phi_{s+t} = \phi_t \circ \phi_s$  for all  $s, t \in [0, \infty)$ ,
3.  $\lim_{t \rightarrow s} \phi_t(z) = \phi_s(z)$  for all  $s \in [0, \infty)$  and  $z \in \mathbb{D}$ ,
4.  $\lim_{t \rightarrow 0^+} \phi_t(z) = z$  locally uniformly on  $\mathbb{D}$ .

In the definition, only right continuity at 0 is required.

The following theorem is fundamental in the theory of one-parameter semigroups.

**Theorem 4.1.** *Let  $\{\phi_t\}_{t \geq 0}$  be a one-parameter semigroup of holomorphic self-mappings of  $\mathbb{D}$ . Then for each  $z \in \mathbb{D}$  there exists the limit*

$$\lim_{t \rightarrow 0^+} \frac{\phi_t(z) - z}{t} =: G(z) \quad (4.1)$$

which  $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$ . The convergence in (4.1) is uniform on each compact subset of  $\mathbb{D}$ .

Conversely, for a function  $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$ , the Cauchy problem

$$\frac{d\phi_t(z)}{dt} = G(\phi_t(z)) \quad (t \geq 0)$$

with the initial condition  $\phi_0(z) = z$  has a unique solution and is a one-parameter semigroup of holomorphic maps of  $\mathbb{D}$ .

The above function  $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$  is called the **infinitesimal generator** of the semigroup. Various criteria which guarantee that a holomorphic function  $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$  is the infinitesimal generator are known. As one of them, in 1978 Berkson and Porta gave the following fundamental characterization.

**Theorem 4.2** ([BP78]). *A holomorphic function  $G \in \text{Hol}(\mathbb{D}, \mathbb{C})$  is the infinitesimal generator if and only if there exists a  $\tau \in \overline{\mathbb{D}}$  and a function  $p \in \text{Hol}(\mathbb{D}, \mathbb{C})$  with  $\text{Re } p(z) \geq 0$  for all  $z \in \mathbb{D}$  such that*

$$G(z) = (\tau - z)(1 - \bar{\tau}z)p(z) \quad (4.2)$$

for all  $z \in \mathbb{D}$ .

The equation (4.2) is called the **Berkson-Porta representation**. In fact, the point  $\tau$  in (4.2) is the Denjoy-Wolff point of the one-parameter semigroup generated with  $G$ .

**4.2. Generalized evolution families in the unit disk.** We have discussed in Section 3.1 that a Loewner chain  $f_t$  (in the classical sense) defines a function  $\varphi_{s,t} := f_t^{-1} \circ f_s : \mathbb{D} \rightarrow \mathbb{D}$  which is called an evolution family. Recently, this notion and one-parameter semigroups are unified and generalized as following. Here  $L^d(X, Y)$  is the family of measurable functions  $f : X \rightarrow Y$  such that  $(\int_X f^d du)^{1/d} < \infty$ .

**Definition 4.3** ([BCDM12, Definition 3.1]). A family of holomorphic self-maps of the unit disk  $(\varphi_{s,t})$ ,  $0 \leq s \leq t < \infty$ , is an **evolution family of order  $d$** , or in short an  **$L^d$ -evolution family**, with  $d \in [1, \infty]$  if

- EF1.  $\varphi_{s,s}(z) = z$ ,
- EF2.  $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$  for all  $0 \leq s \leq u \leq t < \infty$ ,
- EF3. for all  $z \in \mathbb{D}$  and for all  $T > 0$  there exists a non-negative function  $k_{z,T} \in L^d([0, T], \mathbb{R})$  such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi$$

for all  $0 \leq s \leq u \leq t \leq T$ .

We denote a family of  $L^d$ -evolution families by  $\mathbf{EF}^d$ . Compared with the case of evolution families and one-parameter semigroups, there is no guarantee that a family  $(\varphi_{s,t}) \in \mathbf{EF}^d$  has a common fixed point in  $\overline{\mathbb{D}}$ .

Some fundamental properties of  $\mathbf{EF}^d$  are derived as follows.

**Theorem 4.4** ([BCDM12, Proposition 3.7, Corollary 6.3]). *Let  $(\varphi_{s,t}) \in \mathbf{EF}^d$ .*

- (i)  $\varphi_{s,t}$  is univalent in  $\mathbb{D}$  for all  $0 \leq s \leq t < \infty$ .
- (ii) For each  $z_0 \in \mathbb{D}$  and  $s_0 \in [0, \infty)$ ,  $\varphi_{s_0,t}(z_0)$  is locally absolutely continuous on  $t \in [s_0, \infty)$ .
- (iii) For each  $z_0 \in \mathbb{D}$  and  $t_0 \in (0, \infty)$ ,  $\varphi_{s,t_0}(z_0)$  is absolutely continuous on  $s \in [0, t_0]$ .

Next, we extend the notion of Infinitesimal generators to the same structure as  $L^d$ -evolution families.

**Definition 4.5** ([BCDM12, Definition 4.1, Definition 4.3]). **A weak holomorphic vector field of order  $d \in [1, \infty]$**  on the unit disk  $\mathbb{D}$  is a function  $G : \mathbb{D} \times [0, \infty) \rightarrow \mathbb{C}$  with the following properties:

- WV1. For all  $z_0 \in \mathbb{D}$ , the function  $G(z_0, t)$  is measurable on  $t \in [0, \infty)$ ,
- WV2. For all  $t_0 \in [0, \infty)$ , the function  $G(z, t_0)$  is holomorphic on  $\mathbb{D}$ ,
- WV3. For any compact set  $K \subset \mathbb{D}$  and for all  $T > 0$ , there exists a non-negative function  $k_{K,T} \in L^d([0, T], \mathbb{R})$  such that

$$|G(z, t)| \leq k_{K,T}(t)$$

for all  $z \in K$  and for almost every  $t \in [0, T]$ .

Furthermore,  $G$  is said to be a **Herglotz vector field of order  $d$**  and denoted by  $G \in \mathbf{HV}^d$  if  $G(\cdot, t)$  is the infinitesimal generator of a semigroup of holomorphic functions for almost all  $t \in [0, \infty)$ .

The following theorem states the relation between  $(\varphi_{s,t}) \in \mathbf{EF}^d$  and  $G \in \mathbf{HV}^d$ . In what follows, an **essential unique**  $f(x)$  means if there exists another function  $g(x)$  which satisfies the statement then  $f(x) = g(x)$  for almost all  $x$ .

**Theorem 4.A** ([BCDM12, Theorem 5.2, Theorem 6.2]). *Let  $d \in [1, \infty]$  be fixed. Then, for any  $(\varphi_{s,t}) \in \mathbf{EF}^d$ , there exists an essential unique  $G \in \mathbf{HV}^d$  such that*

$$\frac{d\varphi_{s,t}(z)}{dt} = G(\varphi_{s,t}(z), t) \tag{4.3}$$

for all  $z \in \mathbb{D}$  and almost all  $t \in [0, \infty)$ . Conversely, for any  $G \in \mathbf{HV}^d$ , a unique solution of (4.3) with the initial condition  $\varphi_{s,s}(z) = z$  is an evolution family of order  $d$ .

The similar mutual characterization holds between a Herglotz vector field of order  $d$  and a pair of the generalized Denjoy-Wolff point  $\tau$  and a generalized Herglotz function.

**Definition 4.6** ([BCDM12, Definition 4.5]). **A Herglotz function of order  $d \in [1, \infty]$**  on the unit disk  $\mathbb{D}$  is a function  $p : \mathbb{D} \times [0, \infty) \rightarrow \mathbb{C}$  with the following properties:

- HF1. For all  $z_0 \in \mathbb{D}$ , the function  $p(z_0, t)$  belongs to  $L^d_{\text{loc}}([0, \infty), \mathbb{C})$  on  $t \in [0, \infty)$ ,
- HF2. For all  $t_0 \in [0, \infty)$ , the function  $G(z, t_0)$  is holomorphic on  $\mathbb{D}$ ,
- HF3.  $\text{Re } p(z, t) \geq 0$  for all  $z \in \mathbb{D}$  and  $t \in [0, \infty)$ .

We denote  $\mathbf{HF}^d$  a family of all Herglotz functions of order  $d$ .

**Theorem 4.7** ([BCDM12, Theorem 4.8]). *Let  $G \in \mathbf{HV}^d$ . Then there exists an essential unique measurable function  $\tau : [0, \infty) \rightarrow \overline{\mathbb{D}}$  and  $p \in \mathbf{HF}^d$  such that*

$$G(z, t) = (\tau(t) - z)(1 - \overline{\tau(t)}z)p(z, t) \quad (4.4)$$

*for all  $z \in \mathbb{D}$  and almost all  $z \in [0, \infty)$ . Conversely, for a given measurable function  $\tau : [0, \infty) \rightarrow \overline{\mathbb{D}}$  and  $p \in \mathbf{HF}^d$ , the equation (4.4) forms a Herglotz vector field of order  $d$ .*

For convenience, we call the above measurable function  $\tau : [0, \infty) \rightarrow \overline{\mathbb{D}}$  the **Denjoy-Wolff function** and denote by  $\tau \in \mathbf{DW}$ . A pair  $(p, \tau)$  of  $p \in \mathbf{HV}^d$  and  $\tau \in \mathbf{DW}$  is called the **Berkson-Porta data** for  $G \in \mathbf{HV}^d$ . We denote the set of all Berkson-Porta datas by  $\mathbf{BP}$ . Hence, there is an essentially one-to-one correspondence among  $(\varphi_{s,t}) \in \mathbf{EF}^d$ ,  $G \in \mathbf{HV}^d$  and  $(p, \tau) \in \mathbf{BP}$ . In particular, the relation of  $\varphi_{s,t}$  and  $(p, \tau)$  is described by the ordinary differential equation

$$\dot{\varphi}_{s,t}(z) = (\tau(t) - \varphi_{s,t}(z))(1 - \overline{\tau(t)}\varphi_{s,t}(z))p(\varphi_{s,t}(z), t) \quad (4.5)$$

which incorporates the Loewner-Kufarev ODE (3.2) and the Berkson-Porta representation (4.2) as special cases.

**4.3. Generalized Loewner chains.** According to the notion of  $L^d$ -evolution families, Loewner chains are also generalized as following.

**Definition 4.8** ([CDMG10, Definition 1.2]). A family of holomorphic functions  $(f_t)_{t \geq 0}$  on the unit disk  $\mathbb{D}$  is called a **Loewner chain of order  $d$**  with  $d \in [1, \infty]$ , or in short an  **$L^d$ -Loewner chain**, if

- LC1.  $f_t : \mathbb{D} \rightarrow \mathbb{C}$  is univalent for each  $t \in [0, \infty)$ ,
- LC2.  $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$  for all  $0 \leq s < t < \infty$ ,
- LC3. for any compact set  $K \subset \mathbb{D}$  and all  $T > 0$ , there exists a non-negative function  $k_{K,T} \in L^d([0, T], \mathbb{R})$  such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi) d\xi$$

for all  $z \in K$  and all  $0 \leq s \leq t \leq T$ .

Further, a Loewner chain of order  $d$  will be said to be **normalized** if  $f_0 \in \mathcal{S}$ .

We denote a family of  $L^d$ -Loewner chains by  $\mathbf{LC}^d$  and normalized  $L^d$ -Loewner chains by  $\mathbf{LC}_0^d$ . Under this definition, a classical Loewner chain given in Section 3 is recognized as an  $L^\infty$ -Loewner chain. We remark that in Definition 4.8, any assumption is not required to  $f_t(0)$  and  $f'_t(0)$ . It implies that a subordination property that  $f_s(\mathbb{D}_r) \subset f_t(\mathbb{D}_r)$  for all  $r \in (0, 1)$  and  $0 \leq s < t < \infty$  does not hold any longer in general. Further, we even do not know whether the **Loewner range**

$$\Omega[(f_t)] := \bigcup_{t \geq 0} f_t(\mathbb{D})$$

is the whole complex plane or not. One of the simplest example of  $L^d$ -Loewner chains is  $f_t(z) := ze^{it}$ . In the classical setting it cannot be a Loewner chain because by Koebe's 1/4-Theorem  $f_s(\mathbb{D})$  is not allowed to be the same as  $f_t(\mathbb{D})$  for any  $s < t$ .

The next theorem gives a relation between  $L^d$ -Loewner chains and  $L^d$ -evolution families.



**Theorem 4.9** ([CDMG10, Theorem 1.3]). *Let  $d \in [1, \infty]$ . For any  $(f_t) \in \text{LC}^d$ , if we define*

$$\varphi_{s,t}(z) := (f_t^{-1} \circ f_s)(z) \quad (z \in \mathbb{D}, 0 \leq s \leq t < \infty)$$

*then  $(\varphi_{s,t}) \in \text{EF}^d$ . Conversely, for any  $(\varphi_{s,t}) \in \text{EF}^d$ , there exists a  $(f_t) \in \text{LC}^d$  such that the following equality holds*

$$(f_t \circ \varphi_{s,t})(z) = f_s(z) \quad (z \in \mathbb{D}, 0 \leq s \leq t < \infty). \quad (4.6)$$

Differentiate both sides of (4.6) with respect to  $t$  then  $f'_t(\varphi_{s,t}) \cdot \dot{\varphi}_{s,t} + \dot{f}_t(\varphi_{s,t}) = 0$  and therefore combining to (4.5) we have the following generalized Loewner-Kufarev PDE

$$\dot{f}_t(z) = (z - \tau(t))(1 - \overline{\tau(t)}z)f'_t(z)p(z, t). \quad (4.7)$$

We shall observe (4.7). Since the term  $\dot{f}_t(z)$  gives a velocity vector at the point  $f_t(z)$ , the right-hand side of the equation (4.7) defines a vector field on  $f_t(\mathbb{D})$ . Assume that  $p$  is not identically zero. Then  $\dot{f}_t(z) = 0$  if  $z = \tau(t)$ . It implies that the point  $f_t(\tau(t))$  plays a role of an "eye" of the flow described by  $f_t(z)$ . Since the Denjoy-Wolff function  $\tau$  is assumed to be only measurable w.r.t.  $t$ , the origin  $f_t(\tau(t))$  of the vector field moves measurably. This observation indicates that  $L^d$ -Loewner chain describes various flows of expanding simply-connected domains. The classical (radial) Loewner-Kufarev PDE is given as the special case of (4.7) with  $\tau \equiv 0$ .

In general, for a given  $L^d$ -evolution family  $(\varphi_{s,t})$ , the equation (4.6) does not define a unique  $L^d$ -Loewner chain. That is, there is no guarantee that  $\mathcal{L}[(\varphi_{s,t})]$ , a family of normalized  $L^d$ -Loewner chains associated with  $(\varphi_{s,t}) \in \text{EF}^d$ , consists of one function. However,  $\mathcal{L}[(\varphi_{s,t})]$  always includes one special  $L^d$ -Loewner chain (in [CDMG10], such a chain is called to be **standard**) and in this sense  $(f_t)$  is determined uniquely. Further, it is sometimes an only member of  $\mathcal{L}[(\varphi_{s,t})]$ . The following theorem states such properties of the uniqueness for  $L^d$ -Loewner chains.

**Theorem 4.10** ([CDMG10, Theorem 1.6 and Theorem 1.7]). *Let  $(\varphi_{s,t}) \in \text{EF}^d$ . Then there exists a unique  $(f_t) \in \text{LC}^d$  such that  $\Omega[(f_t)]$  is either  $\mathbb{C}$  or an Euclidean disk in  $\mathbb{C}$  whose center is the origin. Furthermore;*

- *The following 4 statements are equivalent;*
  - (i)  $\Omega[(f_t)] = \mathbb{C}$ ,
  - (ii)  $\mathcal{L}[(\varphi_{s,t})]$  consists of only one function,
  - (iii)  $\beta(z) = 0$  for all  $z \in \mathbb{D}$ , where

$$\beta(z) := \lim_{t \rightarrow +\infty} \frac{|\varphi'_{0,t}(z)|}{1 - |\varphi_{0,t}(z)|^2},$$

- (iv) *there exists at least one point  $z_0 \in \mathbb{D}$  such that  $\beta(z_0) = 0$ .*

- *On the other hand, if  $\Omega[(f_t)] \neq \mathbb{C}$ , then it is written by*

$$\Omega[(f_t)] = \left\{ w : |w| < \frac{1}{\beta(0)} \right\},$$

*and for the other normalized Loewner chain  $g_t$  associated with  $(\varphi_{s,t})$ , there exists  $h \in \mathcal{S}$  such that*

$$g_t(z) = \frac{h(\beta(0)f_t(z))}{\beta(0)}.$$

**4.4. Quasiconformal extensions for  $L^d$ -Loewner chains of radial type.** In view of Theorem 3.4, a simple question is proposed that whether the same assumption for  $p \in \mathbf{HF}^d$  that  $p \in U(k)$  deduces quasiconformal extensibility of the corresponding  $(f_t) \in \mathbf{LC}^d$  or not. We give a positive answer to this problem under the special case that  $\tau \in \mathbf{DW}$  is constant. According to the case that  $\tau \in \mathbb{D}$  or  $\tau \in \partial\mathbb{D}$ , the corresponding setting is called the **radial case** or **chordal case**. In the classical Loewner theory, the first is the original case introduced by Löwner, and the second is considered firstly by Kufarev et al. [KSS68].

We employ the following definition due to [CDMG].

**Definition 4.11** ([CDMG, Definition 1.2]). Let  $(\varphi_{s,t}) \in \mathbf{EF}^d$ . Suppose that all non-identical elements of  $(\varphi_{s,t})$  share the same point  $\tau_0 \in \overline{\mathbb{D}}$  such that  $\varphi_{s,t}(\tau_0) = \tau_0$  and  $|\varphi'_{s,t}(\tau_0)| \leq 1$  for all  $s \geq 0$  and  $t \geq s$ , where  $\varphi_{s,t}(\tau_0)$  and  $\varphi'_{s,t}(\tau_0)$  are to be understood as the corresponding angular limit if  $\tau_0 \in \partial\mathbb{D}$ . Then  $\varphi_{s,t}$  is said to be a **radial evolution family** if  $\tau_0 \in \mathbb{D}$ , or a **chordal evolution family** if  $\tau_0 \in \partial\mathbb{D}$ .

Then the radial and chordal version of Loewner chains are defined.

**Definition 4.12** ([CDMG, Definition 1.5]). Let  $(f_t) \in \mathbf{LC}^d$ . If  $(\varphi_{s,t})_{0 \leq s \leq t < \infty} := (f_t^{-1} \circ f_s)_{0 \leq s \leq t < \infty}$  is a radial (or chordal) evolution family of order  $d$ , then we call  $(f_t)$  a  $(L^d)$ -**Loewner chain of radial (or chordal) type**.

Firstly we discuss quasiconformal extensions for  $L^d$ -Loewner chains of radial type.

**Theorem 4.13.** *Let  $k \in [0, 1)$  be a constant. Suppose that  $(f_t)$  is an  $L^d$ -Loewner chain of radial type for which  $p \in \mathbf{HF}^d$  in (4.7) satisfies*

$$p(z, t) \in U(k)$$

*for all  $z \in \mathbb{D}$  and almost all  $t \geq 0$  and  $\tau \in \mathbf{DW}$  is equal to 0. Then the following assertions hold;*

- (i)  $f_t$  admits a continuous extension to  $\overline{\mathbb{D}}$  for each  $t \geq 0$ ,
- (ii)  $F$  defined in (3.5) gives a  $k$ -quasiconformal extension of  $f_0$  to  $\mathbb{C}$ ,
- (iii)  $\Omega[(f_t)] = \mathbb{C}$ .

*Proof.* With no loss of generality, we may assume  $(f_t) \in \mathbf{LC}_0^d$ . Let  $\rho \in (c, 1)$  with some constant  $c \in (0, 1)$  and define  $f_t^\rho(z) := f_t(\rho z)/\rho$ . Then accordingly  $F_\rho$  is defined. Since  $f_t^\rho(z)$  satisfies  $\partial_t f_t^\rho(z) := z \partial_z f_t^\rho(z) p(\rho z, t)$ ,  $f_t^\rho$  satisfies all the assumption of our theorem. Further,  $f_t^\rho$  is well-defined on  $\overline{\mathbb{D}}$ .

Take two points  $z_1, z_2 \in \mathbb{C}$ ,  $z_1 \neq z_2$ . If either  $z_1$  or  $z_2$  is in  $\mathbb{D}$ , then it is clear that  $F_\rho(z_1) \neq F_\rho(z_2)$ . Suppose  $z_1 := r_1 e^{i\theta_1}$ ,  $z_2 := r_2 e^{i\theta_2} \in \mathbb{C} \setminus \mathbb{D}$  such that  $F_\rho(z_1) = F_\rho(z_2)$ , namely  $f_{\log r_1}(\rho e^{i\theta_1}) = f_{\log r_2}(\rho e^{i\theta_2})$ . Denote  $t_1 := \log r_1$  and  $t_2 := \log r_2$ . Since  $f_t^\rho(\partial\mathbb{D})$  is Jordan,  $t_1 \neq t_2$ . By the equality condition of the Schwarz lemma we have  $\varphi_{t_1, t_2}(z) := f_{t_2}^{-1} \circ f_{t_1}(z) = e^{i\theta} z$  for some  $\theta \in \mathbb{R}$ . Hence  $p(\mathbb{D}, t)$  lies on the imaginary axis for all  $t \in [t_1, t_2]$  which contradicts our assumption. We conclude that  $F_\rho$  is a homeomorphism on  $\mathbb{C}$ .

A simple calculation shows that  $|\partial_{\bar{z}} F_\rho(z)/\partial_z F_\rho(z)| = |(\partial_t f_t^\rho(z) - z \partial_z f_t^\rho(z))/(\partial_t f_t^\rho(z) + z \partial_z f_t^\rho(z))| \leq k$ . We conclude that  $F_\rho$  is  $k$ -quasiconformal on  $\mathbb{C}$ . Since the maximal dilatation of  $F_\rho$  does not depend on  $\rho \in (c, 1)$ ,  $(F_\rho)_{\rho \in (c, 1)}$  forms a family of  $k$ -quasiconformal mappings on  $\mathbb{C}$  and is normal. Therefore the limit  $F(z) = \lim_{\rho \rightarrow 1} F_\rho(z)$  exists which gives a  $k$ -quasiconformal extension of  $f_0$ . In particular,  $f_t$  is defined on  $\partial\mathbb{D}$  for all  $t \geq 0$ . It also follows from quasiconformality of  $F$  that  $F(\mathbb{C}) = \Omega[(f_t)] = \mathbb{C}$ .  $\square$

If  $(f_t) \in \text{LC}_0^d$  associated with  $(p, \tau)$  and  $\tau \in \text{DW}$  is an internal fixed point on  $\mathbb{D} \setminus \{0\}$ , then let  $(g_t) \in \text{LC}_0^d$  associated with  $(p, 0) \in \text{BP}$ . Since  $\Omega[(g_t)] = \mathbb{C}$ ,  $(g_t)$  is uniquely determined. Applying the above theorem,  $F$  defines a  $k$ -quasiconformal extension of  $g_0$ . Now, it is verified by Theorem 4.10 that the equation  $f_t(z) = g_t(M(z))$  holds, where

$$M(z) := \frac{z - \tau}{1 - \bar{\tau}z}.$$

In particular  $f_0 = g_0 \circ M$ . Hence  $F \circ M$  gives a  $k$ -quasiconformal extension of  $f_0$ .

**4.5. Quasiconformal extensions for  $L^d$ -Loewner chains of chordal type.** An  $L^d$ -Loewner chain of chordal type with a quasiconformal extension is discussed by Gumenyuk and the author [GH]. In this case by some rotation we can assume that  $\tau \in \text{DW}$  is equal to 1. In their paper, the half-plane model of  $L^d$ -chordal evolution families  $\Phi_{s,t} : \mathbb{H} \rightarrow \mathbb{H}$  and  $L^d$ -Loewner chains of chordal type  $f_t : \mathbb{H} \rightarrow \mathbb{C}$  are treated. In fact, everything can be transferred to the right-half plane by a Cayley map  $K(z) = (1+z)/(1-z)$ . In this case the generalized Loewner-Kufarev PDE and ODE is written by

$$\dot{\Phi}_{s,t}(\zeta) = p_{\mathbb{H}}(\Phi_{s,t}(\zeta), t) \quad \text{and} \quad \dot{f}_t(\zeta) = -f'_t(\zeta)p_{\mathbb{H}}(\zeta, t) \quad (\zeta \in \mathbb{H}), \quad (4.8)$$

where  $p_{\mathbb{H}}$  stands for the right half-plane model of Herglotz function of order  $d$ . A special case of (4.8) that  $p_{\mathbb{H}}(\zeta) = 1/(\zeta + i\lambda(t))$  has attracted great attention since the work by Schramm [Sch00] was provided, where  $\lambda : [0, \infty) \rightarrow \mathbb{R}$  is a measurable function.

The next theorem states the  $L^d$ -chordal version of Becker's theorem.

**Theorem 4.14** ([GH]). *Suppose that a family of holomorphic functions  $(f_t)_{t \geq 0}$  on the right half-plane  $\mathbb{H}$  is an  $(L^d)$ -Loewner chain of chordal type with some  $d \in [1, \infty]$ . If there exists a uniform constant  $k \in [0, 1)$  such that  $p_{\mathbb{H}}$  satisfies*

$$p_{\mathbb{H}}(\zeta, t) \in U(k) \quad (4.9)$$

for all  $\zeta \in \mathbb{H}$  and almost all  $t \geq 0$ , then

- (i)  $f_t$  admits a continuous extension to  $\mathbb{H} \cup i\mathbb{R}$ ,
- (ii)  $f_t$  has a  $k$ -quasiconformal extension to  $\mathbb{C}$  for each  $t \geq 0$ . In this case the extension  $F$  is explicitly given by

$$F(\zeta) := \begin{cases} f_0(\zeta), & \zeta \in \mathbb{H}, \\ f_{-\text{Re}\zeta}(i \text{Im}\zeta), & \zeta \in \mathbb{C} \setminus \overline{\mathbb{H}}, \end{cases}$$

- (iii)  $\Omega[(f_t)] = \mathbb{C}$ .

If  $\tau \in \text{DW}$  is a boundary point on  $\partial\mathbb{D} \setminus \{1\}$ , then composing some rotation we obtain the same result as Theorem 4.14. In fact, by setting  $g_t(z) := f_t(\bar{\tau}z)$  we have  $g_t(z) = (z - \tau)(1 - \bar{\tau}z)g'_t(z)p(\bar{\tau}z, t)$ . After transferring  $g_t$  to the right half-plane, Theorem 4.14 with the same  $k$  as  $f_t$  is applied.

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